The Open University

Mathematics, Computing and Technology
MS221 Exploring Mathematics

MS221

Assignment Booklet I 2012J

Contents | **Cut-off date**

Instructions for submitting assignments

Please send all your answers to each tutor-marked assignment (TMA), together with a completed assignment form (PT3), to reach your tutor on or before the appropriate cut-off date shown above.

Your tutor will inform you about the address to use for submitting your TMAs. Please don't submit your TMAs directly to the University. Regrettably, the University is unable to accept TMAs submitted electronically on this module. If you have any questions about how best to prepare and submit your TMAs, please contact your tutor.

Be sure to fill in the correct assignment number on the PT3 form, and allow sufficient time in the post for each assignment to reach its destination on or before the cut-off date.

You are advised to keep a copy of your assignment in case of loss in the mail. Also, keep all your marked assignments as you may need to make reference to them in later assignments or when you revise for the examination.

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Plagiarism

The University considers plagiarism to be a serious matter, therefore we draw your attention to the appendix on plagiarism in the *Assessment Handbook* entitled 'What constitutes plagiarism or cheating?'. Please note that references to 'assignments' should be taken to include any piece of work submitted for assessment, not just tutor-marked assignments.

Points to note when preparing solutions to TMA questions

- Contact your tutor if the meaning of any part of a question does not seem clear.
- Your solutions should not involve the use of Mathcad, except in those parts of questions where this is explicitly required or suggested.
- Where a question involves mathematical calculation, show all your working. You may not receive full marks for a correct final answer that is not supported by working. You may receive some marks for working even if your final answer is incorrect or your solution is incomplete.
- When you are asked for the exact value of a number, this means an algebraic expression possibly including fractions of integers, surds, the numbers π and e, and so on, rather than a decimal approximation to the number obtained using a calculator.
- In questions other than those in which you are asked to 'write down' an answer, some marks may be reserved for a reasonable explanation of what you are doing.
- Whenever you perform a calculation using a numerical answer found earlier, you should use the full-calculator-precision version of the earlier answer to avoid rounding errors.
- Number all of your pages, including any computer printouts.
- Indicate in each solution the page numbers of any computer printouts associated with that solution.
- The marks allocated to the parts of the questions are indicated in brackets in the margin. Each TMA is marked out of 100. Your overall score for a TMA will be the sum of your marks for each question part.

This assignment covers Block A.

Question 1 – 10 marks

You should be able to answer this question after studying Chapter A1.

Find a closed form for the following sequence:

$$u_0 = 11,\ u_1 = -1, \quad u_{n+2} = \tfrac{13}{5}u_{n+1} + \tfrac{6}{5}u_n \quad (n = 0, 1, 2, \ldots). \qquad [10]$$

Question 2 – 10 marks

You should be able to answer this question after studying Chapter A1.

Let u_n be the sequence

$$u_n = 4^n + (-9)^n \quad (n = 0, 1, 2, \ldots).$$

Show that u_n satisfies the identity

$$u_{n+1}u_{n-1} - u_n^2 = 169(-36)^{n-1}, \quad \text{for } n = 1, 2, 3, \ldots. \qquad [10]$$

Question 3 – 15 marks

You should be able to answer this question after studying Chapter A1.

Let F_n be the Fibonacci sequence.

(a) (i) Use the Fibonacci recurrence relation to express F_{n+3} in terms of F_{n+1} and F_n. Hence show that $F_{n+1} = \frac{1}{2}(F_{n+3} - F_n)$ for $n = 0, 1, 2, \ldots$. [4]

(ii) Use the Fibonacci recurrence relation to express F_{n+3} in terms of F_{n+2} and F_n. Hence show that $F_{n+2} = \frac{1}{2}(F_{n+3} + F_n)$ for $n = 0, 1, 2, \ldots$. [3]

(iii) Would the formulas in parts (a)(i) and (a)(ii) necessarily remain true if the sequence F_n were replaced by a sequence with the same recurrence relation as the Fibonacci sequence but different initial terms? Justify your answer, briefly. [2]

In part (b) you should provide a printout of your tables as part of your solution.

(b) In this part of the question, you are asked to use Mathcad to explore a connection between the numbers

$$F_{n+4}^3 + F_n^3 \quad \text{and} \quad F_{n+3}^3 + 2F_{n+2}^3 - F_{n+1}^3,$$

for $n = 0, 1, 2, \ldots$, as follows.

Use a copy of Mathcad file 221A1-02 to help you to display corresponding values of these numbers side by side. The file uses the notation u_n rather than F_n for the Fibonacci numbers. Edit the Mathcad file to display the sequences

$$n, \quad u_{n+4}^3 + u_n^3, \quad u_{n+3}^3 + 2u_{n+2}^3 - u_{n+1}^3$$

for $n = 1, 2, \ldots, 18$. You will need to change the display range to $n := 1, 2 \,..\, N - 3$ and set N to an appropriate value.

Use your Mathcad printout to make a conjecture connecting $F_{n+4}^3 + F_n^3$ and $F_{n+3}^3 + 2F_{n+2}^3 - F_{n+1}^3$. [6]

Question 4 – 30 marks

You should be able to answer this question after studying Chapter A2.

(a) This part of the question concerns the conic with equation

$$2x^2 + 5y^2 = 50.$$

(i) By rearranging the equation of the conic, classify it as an ellipse, parabola or hyperbola in standard position, and sketch the curve. [6]

(ii) Show that the eccentricity of this conic is $\frac{1}{5}\sqrt{15}$. Hence find the coordinates of the foci and the equations of the directrices, giving the exact values of the numbers involved. Mark the foci and directrices on your sketch. [5]

(iii) For each of the two points P where the conic intersects the x-axis, use the exact values found in part (a)(ii) to calculate the distances PF and Pd, where F is the focus with negative x-coordinate and d is the corresponding directrix. Show that $PF = e\,Pd$, where e is the eccentricity. (This provides a check on your answers to part (a)(ii).) [4]

(b) Now consider the curve with equation

$$\tfrac{2}{3}x^2 + \tfrac{5}{3}y^2 - \tfrac{8}{3}x + 10y + 1 = 0.$$

(i) Show that this curve is a conic that can be obtained from the conic in part (a) by translation, and describe the translation required. [5]

(ii) Use your answers to part (a) to sketch this conic, showing its centre, vertices and axes of symmetry, and the slopes of any asymptotes. Give the exact values of the numbers involved. [4]

(c) (i) Write down parametric equations for the conics in parts (a) and (b). [2]

In part (c)(ii) you should provide a printout showing your plot.

(ii) Use the parametric equations in part (c)(i) to plot both conics on the same diagram, using Mathcad. (You may find it useful to start from Mathcad file 221A2-01.) [4]

Question 5 – 20 marks

You should be able to answer this question after studying Chapter A3.

(a) Using the exact values of the sine and the cosine of $\frac{1}{6}\pi$ and $\frac{3}{4}\pi$, and one of the sum and difference formulas, show that the exact value of $\sin\left(\frac{7}{12}\pi\right)$ is $\frac{1}{4}(\sqrt{2} + \sqrt{6})$. [5]

(b) (i) By using the sum and difference formulas for $\cos(\phi + \theta)$ and $\cos(\phi - \theta)$, with $\phi = \frac{1}{2}(A + B)$ and $\theta = \frac{1}{2}(A - B)$, derive the formula

$$\cos A + \cos B = 2\cos(\tfrac{1}{2}(A + B))\cos(\tfrac{1}{2}(A - B)).$$ [4]

(ii) Derive the formula

$$\cos A - \cos B = -2\sin(\tfrac{1}{2}(A + B))\sin(\tfrac{1}{2}(A - B)).$$ [2]

(You should be able to adapt your solution to part (b)(i).)

(c) Given that $\operatorname{cosec}\theta = 2$, $\cot\theta = -\sqrt{3}$ and $-\pi < \theta < \pi$, find the exact value of the angle θ in radians. Justify your answer. [3]

(d) Given that $\frac{1}{2}\pi < \theta < \pi$ and $\sin\theta = \frac{1}{5}$, use appropriate trigonometric formulas to find the exact values of the following.

(i) $\cos(2\theta)$ [2]

(ii) $\cos\theta$ [2]

(iii) $\sin(2\theta)$ [2]

Question 6 – 15 marks

You should be able to answer this question after studying Chapter A3.

(a) A triangle has vertices at the points $A(6,-2)$, $B(-4,2)$ and $C(4,6)$. Suppose that the triangle is to be moved so that B is at the origin and BA lies along the positive x-axis. One isometry that achieves this transformation is the composite of a translation followed by a rotation. (You may find it helpful to sketch the triangle.)

(i) Determine the translation that moves B to the origin, giving your answer in the form $t_{a,b}$. Write down a formal definition of this translation in two-line notation. [2]

(ii) Find the images A' of A and C' of C under the translation in part (a)(i). [2]

(iii) Let r_θ be the rotation that completes the required transformation, where θ lies in the interval $(-\pi, \pi]$. Find the exact values of $\tan\theta$, $\cos\theta$ and $\sin\theta$, and hence write down a formal definition of r_θ using two-line notation. (There is no need to work out the value of the angle θ.) [5]

(iv) Find the coordinates of the images of A' and C' under the rotation r_θ. Give your answers as exact values. [2]

(v) Write down a formal definition, in two-line notation, of the composite transformation, that is, the result of the translation in part (a)(i) followed by the rotation in part (a)(iii). [2]

In part (b) you should provide a printout showing the graph and the appropriate settings in the worksheet as part of your solution.

(b) In this part of the question, you should use a copy of Mathcad file 221A3-02 to obtain the graph of a transformed hyperbola.

Modify the worksheet so that the graph shows the result of applying the rotation $r_{\pi/3}$ to the hyperbola

$$\frac{x^2}{4} - y^2 = 1.$$ [2]

This assignment covers Block B.

Question 1 – 15 marks

You should be able to answer this question after studying Chapter B1.

In this question, f is the function

$$f(x) = -\tfrac{1}{2}x^2 + \tfrac{1}{3}x + \tfrac{2}{3}.$$

(a) Use algebra to find the fixed points of f, and to classify them as attracting, repelling or indifferent. *[6]*

(b) Use the gradient criterion to determine an interval of attraction for one of the fixed points of f. *[5]*

(c) Find the exact values of the second and third terms of the sequence x_n obtained by iterating f with initial term $x_0 = 2$. (Express your answers as fractions in their lowest terms.) Hence state the long-term behaviour of this sequence, explaining your reasoning. *[4]*

Question 2 – 15 marks

You should be able to answer this question after studying Chapter B1.

In this question, you should use a copy of Mathcad file 221B1-03 to study iteration of the function

$$f(x) = \frac{(x+2)(x+4)(2x-1)}{4(x^2+x+1)}.$$

For parts (a)–(d) you should provide printouts illustrating your answers, including a printout for each fixed point in part (a), and a printout for the 2-cycle in part (b). You should reset the axis limits to $s1 := -4$ *and* $s2 := 6$ *to obtain your printouts.*

(a) Find the fixed points of f, and classify them as attracting, repelling or indifferent. *[6]*

(b) Find the 2-cycle of f, and classify it as attracting, repelling or indifferent. *[2]*

(c) Find the long-term behaviour of the sequence obtained by iterating f with initial term $x_0 = 1$, explaining how you reached your conclusion. *[3]*

(d) Repeat part (c) with the initial term $x_0 = -1$. *[4]*

Question 3 – 5 marks

You should be able to answer this question after studying Chapter B1.

(a) By writing down and solving a suitable equation in k, find an integer k such that

$$(x^3)^{21-k}\left(\frac{1}{x}\right)^k = x^{19},$$

for all non-zero values of x. [2]

(b) Use the Binomial Theorem, together with your answer to part (a), to determine the coefficient of x^{19} in the expansion of

$$\left(2x^3 - \frac{1}{2x}\right)^{21},$$

for $x \neq 0$. [3]

Question 4 – 15 marks

You should be able to answer this question after studying Chapter B2.

Let f be the linear transformation represented by the matrix

$$\mathbf{M} = \begin{pmatrix} 0 & 3 \\ 1 & -1 \end{pmatrix}.$$

(a) State what effect f has on areas, and whether f changes orientation. [2]

(b) Find the matrix that represents the inverse of f. [2]

(c) (i) Use the matrix that you found in part (b) to find the image $f(\mathscr{C})$ of the unit circle $\mathscr{C}$ under f, in the form

$$ax^2 + bxy + cy^2 = d,$$

where a, b, c and d are integers. [5]

(ii) What is the area enclosed by $f(\mathscr{C})$? [1]

For part (d), you should provide a printout of your work.

(d) Using page 5 of Mathcad file 221B2-01, express the matrix $\mathbf{M}$ as a product of matrices of basic linear transformations. [5]

Question 5 – 20 marks

You should be able to answer this question after studying Chapter B2.

In this question, f and g are both affine transformations. The transformation f is reflection in the line $y = -x + 3$, and the transformation g maps the points $(0, 0)$, $(1, 0)$ and $(0, 1)$ to the points $(1, -1)$, $(1, 0)$ and $(2, -1)$, respectively.

(a) Determine g in the form $g(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a}$, where $\mathbf{A}$ is a 2×2 matrix and $\mathbf{a}$ is a vector with two components. *[3]*

(b) Express f as a composite of three transformations: a translation, followed by reflection in a line through the origin, followed by a translation. Hence determine f in the same form as you found g in part (a). *[6]*

You may do parts (c) and (d) in either order. The order may depend on the methods that you choose to use.

(c) Find the images of the points $(0, 0)$, $(1, 0)$ and $(0, 1)$ under the composite affine transformation $g \circ f$ (that is, f followed by g). *[3]*

(d) Find the affine transformation $g \circ f$ in the same form as you found g in part (a). *[3]*

(e) Use your answer to part (d) to show that there is exactly one point (x, y) such that the image of (x, y) under $g \circ f$ is (x, y). State the coordinates of this point. *[3]*

(f) Given that $g \circ f$ is a rotation about the point in part (e), find the angle of rotation. One way to do this is to consider the diagram obtained by plotting one of the points from part (c) and its image under $g \circ f$. *[2]*

Question 6 – 30 marks

You should be able to answer this question after studying Chapter B3.

In this question,

$$\mathbf{A} = \begin{pmatrix} 5 & 7 \\ -2 & -4 \end{pmatrix}.$$

(a) Without reference to Mathcad, find the eigenvalues and eigenlines of $\mathbf{A}$. For each eigenvalue, give an eigenvector with integer components. [*8*]

(b) Express $\mathbf{A}$ in the form $\mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix. You should evaluate $\mathbf{P}$ and $\mathbf{P}^{-1}$. [*4*]

(c) Use part (b) to find $\mathbf{A}^7$. [*4*]

(d) Calculate the second and third points of the iteration sequence with recurrence relation $\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$ $(n = 0, 1, 2, \ldots)$, for each of the following initial points.

(i) $(-1, 1)$

(ii) $(1, 1)$ [*4*]

For part (e), you should provide printouts of your work.

(e) Use page 3 of Mathcad file 221B3-01 to plot the first four points of each of the iteration sequences in part (d). In each case you should set the value of the graph scale variable s to an appropriate number. [*4*]

(f) Describe the long-term behaviour of each of the iteration sequences in part (d). [*6*]